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Application of high-order difference methods for the study of period doubling bifurcations in nonlinear oscillators

Hugo Janssen[a,*], René Van Dooren[b]

[a] *Theoretische Wiskunde, Koninklijke Militaire School, Renaissancelaan 30, B-1040 Brussels, Belgium*
[b] *Werktuigkunde, Vrije Universiteit Brussel, Pleinlaan 2, B-1050 Brussels, Belgium*

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Abstract

For the study of period doubling bifurcations in nonlinear T-periodic forced oscillators depending on one parameter, it is necessary to use a quick and accurate method to obtain periodic solutions of a second-order nonlinear differential equation.

Using a periodic approximation, obtained earlier for another parametric value, a Newton method allows its iterative refinement, by the successive solution of linear periodic boundary value problems. In these problems, the derivatives of the unknown periodic solution are approximated by high-order difference formulae, yielding accurate results in the N abscissae $x_i = iT/N$, $i = 0, \dots, N-1$.

To determine the parametric values where bifurcation takes place, the solution of the variational equations was calculated using a Runge–Kutta–Hůta method. Therefore the values of the corresponding nT-periodic solution ($n = 2^k$, $k \in \mathbb{N}$) were determined in some points inside the discretization intervals using trigonometric interpolation. From the resulting fundamental matrix $\Phi(nT)$, the multipliers are readily determined, allowing the detection of the period doubling bifurcations when the largest multiplier passes through -1.

The method has been applied successfully to study bifurcations leading to chaos in several classical nonlinear oscillators.

Keywords: Continuation algorithm; Difference methods; Duffing systems; Bifurcation; Feigenbaum relation; Chaos

1. Introduction

In the following paragraphs, we will use some results obtained by Urabe [18] in order to study periodic nonlinear differential equations of the type

$$\ddot{x} + c\dot{x} + F(x,t) = 0, \tag{1}$$

* Corresponding author.

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where F is nonlinear in x and periodic in t with period 2π.

In [18–20], Urabe used the Galerkin method or the averaging method to determine approximate solutions of periodic differential systems of the form

$$\dot{x} = F(x, t),$$

where $F: \mathbb{R}^n \times [0, 2\pi] \to \mathbb{R}^n$ is periodic in t and $x \in \mathscr{C}^1([0, 2\pi], \mathbb{R}^n)$. Strassberg [17] and D'Hondt [4] used Newton or Newton-like methods, and Van Dooren [21–23] a harmonic or a Chebyshev balance perturbation method to treat these problems.

In this work we combine the Newton method with a high-order difference method to determine approximate periodic solutions of (1) with sufficient precision to follow up a number of period doubling bifurcations, caused by a varying parameter in the given differential equation. We want to show the universal scaling property of the bifurcation structure of Duffing-like equations.

2. Theoretical background

2.1. Notations

(1) In the sequel, we use the symbol $|\ |$ for the Euclidean vector norm in $\mathbb{R}^n$,

$$\forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n\colon \quad |x| = \sqrt{\sum_{i=1}^{n} x_i^2},$$

and the symbol $\|\ \|$ for the associated matrix norm. We will also use the maximum norm $|\ |_\infty$ of a vector in $\mathbb{R}^n$:

$$\forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n\colon \quad |x|_\infty = \max\{|x_i| \mid i \in \{1, \ldots, n\}\}.$$

(2) If $f: [0, 2\pi] \to \mathbb{R}^n : t \mapsto f(t)$ is bounded, then we define

$$\|f\| = \sup\{|f(t)| \mid t \in [0, 2\pi]\}.$$

(3) If h is a piecewise continuous real function in two real variables, and if $x \in \mathscr{C}([0, 2\pi], \mathbb{R})$ and $y \in \mathscr{C}^1([0, 2\pi], \mathbb{R})$, h, x and y being 2π-periodic in all variables, and such that

$$y(t) = \int_0^{2\pi} h(t, s)\, x(s)\, \mathrm{d}s,$$

then this relationship defines a linear map $\mathscr{H}_1$ of $\mathscr{C}([0, 2\pi], \mathbb{R}) \to \mathscr{C}^1([0, 2\pi], \mathbb{R})$, for which the norm associated with $\|\ \|$, due to the Schwarz inequality, satisfies

$$\|\mathscr{H}_1\| \leqslant \sqrt{2\pi \sup_{t \in [0, 2\pi]} \int_0^{2\pi} h^2(t, s)\, \mathrm{d}s}.$$

2.2. Basic theorems

Let us adapt some results of Urabe [18] to the case of the differential equation (1):

Proposition 1. *Let*

$$\mathscr{L}(x) \equiv \ddot{x} + c\dot{x} + a(t)x = f(t) \tag{2}$$

be a given linear periodic differential equation, with a and f continuous 2π-periodic functions. If the multipliers of the corresponding homogeneous differential equation

$$\mathscr{L}(x) = 0 \tag{3}$$

are all different from one, then (2) *has a unique periodic solution of period 2π, given by*

$$x(t) = \int_0^{2\pi} H_{1,2}(t,s) f(s)\,\mathrm{d}s,$$

where $H_{1,2}(t,s)$ is the second element of the first row of a piecewise continuous periodic matrix:

$$H(t,s) = \begin{cases} \Phi(t)(E - \Phi(2\pi))^{-1}\Phi^{-1}(s) & s < t, \\ \Phi(t)(E - \Phi(2\pi))^{-1}\Phi(2\pi)\Phi^{-1}(s) & s \geqslant t. \end{cases} \tag{4}$$

Here E is the unit matrix in $\mathbb{R}^{2\times 2}$ and Φ is the fundamental matrix of (3) *with $\Phi(0) = E$.*

Proof. Putting $x(t) = x_1(t)$, $\dot{x}(t) = x_2(t)$, $X = \binom{x_1}{x_2}$ allows us to write (2) in the form $\dot{X} = A(t)X + F(t)$, where

$$A(t) = \begin{pmatrix} 0 & 1 \\ -a(t) & -c \end{pmatrix}, \quad F(t) = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}.$$

Applying the results of Proposition 1 of Urabe [18] to this system, for which

$$\Phi(t) = \begin{pmatrix} \varphi_1(t) & \varphi_2(t) \\ \dot{\varphi}_1(t) & \dot{\varphi}_2(t) \end{pmatrix} = (\phi_1(t) \quad \phi_2(t))$$

and $\dot{\phi}_i = A\phi_i$, $i \in \{1,2\}$, $\phi_1(0) = \binom{1}{0}$, $\phi_2(0) = \binom{0}{1}$, we obtain a unique periodic solution

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \int_0^{2\pi} H(t,s) \begin{pmatrix} 0 \\ f(s) \end{pmatrix} \mathrm{d}s.$$

Taking the first component of this vector equation yields the desired formula:

$$x_1(t) = x(t) = \int_0^{2\pi} H_{1,2}(t,s) f(s)\,\mathrm{d}s. \qquad \Box \tag{5}$$

Thus we see that the map

$$\mathscr{H}_1 : \mathscr{C}([0,2\pi],\mathbb{R}) \to \mathscr{C}^2([0,2\pi],\mathbb{R}) : f \mapsto x = \mathscr{H}_1(f)$$

given by (5) is the inverse of the map

$$\mathscr{L} : \mathscr{C}^2([0,2\pi],\mathbb{R}) \to \mathscr{C}([0,2\pi],\mathbb{R}) : x \mapsto f = \mathscr{L}(x)$$

given by (2). In the sequel, we will also use the next proposition in [18]:

Proposition 2. *Let*

$$F(\alpha) = 0 \tag{6}$$

be a given system of equations, where $\alpha \in \mathbb{R}^n$ *and* $F: \mathbb{R}^n \to \mathbb{R}^n$ *with* F *continuously differentiable in some region* $\Omega \in \mathbb{R}^n$. *Assume that* (6) *has an approximate solution* $\bar{\alpha} \in \mathbb{R}^n$ *for which the determinant of the Jacobian matrix* $J(\bar{\alpha})$ *of* F *does not vanish, and there are constants* $\delta \in \mathbb{R}_0^+$ *and* $\kappa \in [0, 1[$ *such that for each* α *in the closed ball* $\Omega_\delta = \{\alpha \mid |\alpha - \bar{\alpha}| \leqslant \delta\}$ *lying in* Ω, *one has*

$$\| J(\alpha) - J(\bar{\alpha}) \leqslant \frac{\kappa}{M'}, \quad \forall \alpha \in \Omega_\delta, \tag{7}$$

$$\frac{M'r}{1-\kappa} \leqslant \delta,$$

where r *and* M' *are numbers such that*

$$|F(\bar{\alpha})| \leqslant r, \quad \| J^{-1}(\bar{\alpha}) \| \leqslant M'.$$

Then system (6) *has a unique solution* $\hat{\alpha} \in \Omega_\delta$ *and*

$$|\hat{\alpha} - \bar{\alpha}| \leqslant \frac{M'r}{1-\kappa}. \tag{8}$$

In the applications it is possible to evaluate M', r and the left member of condition (7) as a known increasing function Δ of δ. This allows the analysis of the system of inequalities

$$\Delta(\delta) \leqslant \frac{\kappa}{M'}, \tag{9}$$

$$\frac{M'r}{1-\kappa} \leqslant \delta \tag{10}$$

in order to find suitable values of κ and δ, which not only ensures the existence of an exact solution of (6), but also gives us the error bound (8).

Proposition 3. *Let* $F: \mathbb{R}^2 \to \mathbb{R}: (x,t) \mapsto F(x,t)$ *be* 2π-*periodic in* t *and continuously differentiable in a region* $D \times \mathbb{R} \subset \mathbb{R}^2$, *and*

$$\ddot{x} + c\dot{x} + F(x,t) = 0 \tag{11}$$

be a given differential equation in the unknown function $x: \mathbb{R} \to \mathbb{R}: t \mapsto x(t)$.

Assume that (11) *has a periodic approximate solution* $\bar{x}(t)$ *lying in* D *and that there is a continuous periodic function* $a: \mathbb{R} \to \mathbb{R}$ *and there are constants* $\delta \in \mathbb{R}_0^+$ *and* $\kappa \in [0, 1[$ *such that:*

(1) *The multipliers of the linear homogeneous differential equation*

$$\ddot{y} + c\dot{y} + a(t)y = 0$$

are all different from one, and

(2) $\forall t \in \mathbb{R}: \forall x \in D_\delta: |(\partial F/\partial x)(x,t) - a(t)| \leqslant \kappa/M_1$,
(3) $M_1 r/(1-\kappa) \leqslant \delta$,

where

$$D_\delta = \{x \mid |x - \bar{x}(t)| \leqslant \delta \text{ for some } t \in \mathbb{R}\} \subset D,$$

M_1 is a constant such that

$$M_1 \geqslant \|\mathscr{H}_1\| = \sup_{t \in [0, 2\pi]} \left\{ \left| \int_0^{2\pi} H_{1,2}(t,s) f(s)\, \mathrm{d}s \right| \mid \|f\| \leqslant 1 \right\}$$

and $\mathscr{H}_1$ is the mapping defined by (5) corresponding to $a(t)$,

$r \in \mathbb{R}^+$ *is such that* $\|\ddot{\bar{x}} + c\dot{\bar{x}} + F(\bar{x}, t)\| \leqslant r$.

Then Eq. (11) *has a unique isolated periodic solution $\hat{x}(t)$ in D_δ. Furthermore, we have the error estimate*

$$\|\hat{x} - \bar{x}\| \leqslant \frac{M_1 r}{1 - \kappa}. \tag{12}$$

Proof. As we follow the same lines as in Proposition 3 of Urabe [18], we only give a sketch of the proof in the particular case of Eq. (1). Let us put

$$\eta(t) = \ddot{\bar{x}} + c\dot{\bar{x}} + F(\bar{x}, t).$$

This can be rewritten as

$$\mathscr{L}(\bar{x}) \equiv \ddot{\bar{x}} + c\dot{\bar{x}} + a(t)\bar{x}(t) = \eta(t) + a(t)\bar{x}(t) - F(\bar{x}, t). \tag{13}$$

Since $\bar{x}$ is 2π-periodic, by Proposition 1 we have

$$\bar{x}(t) = \int_0^{2\pi} H_{1,2}(t,s)[\eta(s) + a(s)\bar{x}(s) - F(\bar{x}(s), s)]\, \mathrm{d}s$$

where $H_{1,2}$ was defined in (4). Let us consider the iterative process

$$x^{(k+1)}(t) = \mathscr{H}(x^{(k)}) \equiv \int_0^{2\pi} H_{1,2}(t,s)[a(s)x^{(k)}(s) - F(x^{(k)}(s), s)]\, \mathrm{d}s, \tag{14}$$

where $k \in \mathbb{N}$, $x^{(0)} = \bar{x}$.

(1) Then we prove inductively that (14) can be continued indefinitely, and that $\forall k \in \mathbb{N}$:

$$\|x^{(k+1)} - x^{(k)}\| \leqslant \kappa^k \|x^{(1)} - x^{(0)}\|, \tag{15}$$

$$\|x^{(k+1)} - x^{(0)}\| \leqslant \frac{M_1 r}{1 - \kappa} \leqslant \delta. \tag{16}$$

(2) Moreover, $x^{(k)}$ is a Cauchy sequence, as from formula (15) it easily follows that

$$\forall \varepsilon \in \mathbb{R}_0^+ \colon \; \exists N \in \mathbb{N} \colon \; \forall q \geqslant p \colon \; \|x^{(q)} - x^{(p)}\| \leqslant \frac{\kappa^N}{1 - \kappa} \|x^{(1)} - x^{(0)}\| < \varepsilon$$

for a sufficiently large N. As the space $\mathscr{C}([0, 2\pi], \mathbb{R})$ is complete, this sequence converges to a periodic continuous function $\hat{x} = \lim_{k \to \infty} x^{(k)}$ lying in D_δ.

(3) Due to (14) we can then prove that

$$\left\| \hat{x}(t) - \int_0^{2\pi} H_{1,2}(t,s)[a(s)\hat{x}(s) - F(\hat{x}(s),s)]\, \mathrm{d}s \right\| \leqslant \|\hat{x} - x^{(k+1)}\| + \kappa \|x^{(k)} - \hat{x}\|.$$

Letting $k \to \infty$ we see that

$$\hat{x}(t) = \int_0^{2\pi} H_{1,2}(t,s)[a(s)\hat{x}(s) - F(\hat{x}(s),s)]\, \mathrm{d}s, \tag{17}$$

implying that

$$\mathscr{L}(\hat{x}) \equiv \ddot{\hat{x}} + c\dot{\hat{x}} + a(t)\hat{x}(t) = a(t)\hat{x}(t) - F[(\hat{x}(t),t]$$

or

$$\ddot{\hat{x}} + c\dot{\hat{x}} + F[\hat{x}(t),t] = 0.$$

This says that $\hat{x}$ is a periodic solution of the differential equation (11). Eq. (12) follows from (16) if we let $k \to \infty$.

(a) The *uniqueness of this solution in* D_δ can be proved by contradiction: Let $\hat{x}'$ be a different periodic solution to the given differential equation, then like $\hat{x}$, it also satisfies

$$\hat{x}'(t) = \int_0^{2\pi} H_{1,2}(t,s)[a(s)\hat{x}'(s) - F(\hat{x}'(s),s)]\, \mathrm{d}s.$$

Subtraction of this equation from (17), and taking norms yields

$$\|\hat{x} - \hat{x}'\| \leqslant \kappa \|\hat{x} - \hat{x}'\|$$

so that $\|\hat{x} - \hat{x}'\| = 0$, as $\kappa < 1$.

(b) Finally, putting $\hat{a}(t) = (\partial F/\partial x)(\hat{x}(t),t)$ we can show that each solution y to the linear homogeneous problem

$$\ddot{y} + c\dot{y} + \hat{a}(t)y = 0 \tag{18}$$

must be zero. Indeed, writing the previous equation in the form

$$\ddot{y} + c\dot{y} + a(t)y = [a(t) - \hat{a}(t)]\,y,$$

any periodic solution satisfies

$$y(t) = \int_0^{2\pi} H_{1,2}(t,s)[a(s) - \hat{a}(s)]\,y(s)\, \mathrm{d}s.$$

Taking norms and using hypothesis 2, we get the estimate

$$\|y\| \leqslant M_1 \cdot \frac{\kappa}{M_1} \cdot \|y\| = \kappa \|y\|,$$

implying that y must be the zero function since $\kappa < 1$. As there is no nontrivial solution of (18), all its multipliers must be different from one, and $\hat{x}$ must be an *isolated periodic solution of* (11). □

3. Description of the method

3.1. Determination of a periodic solution of the given differential equation

In order to solve (11) we want to determine a discrete solution $x_i \equiv x(t_i)$ on the grid $\bar{\omega}_h = \{t_i = (i-1)h,\ h = T/N,\ i = 1, \dots, N\}$, using a collocation method:

$$\ddot{x}_i + c\dot{x}_i + F(x_i, t_i) = 0 \quad \text{for } i \in \{1, \dots, N\}. \tag{19}$$

We will evaluate the derivatives $\ddot{x}_i$ and $\dot{x}_i$ using 9-point symmetric differentiation formulae:

$$\dot{x}_i = \frac{(\frac{1}{280}x_{i-4} - \frac{4}{105}x_{i-3} + \frac{1}{5}x_{i-2} - \frac{4}{5}x_{i-1} + \frac{4}{5}x_{i+1} - \frac{1}{5}x_{i+2} + \frac{4}{105}x_{i+3} - \frac{1}{280}x_{i+4})}{h} + \frac{h^8 D_i^9}{630}, \tag{20}$$

$$\ddot{x}_i = \frac{(-\frac{1}{560}x_{i-4} + \frac{8}{315}x_{i-3} - \frac{1}{5}x_{i-2} + \frac{8}{5}x_{i-1} - \frac{205}{72}x_i + \frac{8}{5}x_{i+1} - \frac{1}{5}x_{i+2} + \frac{8}{315}x_{i+3} - \frac{1}{560}x_{i+4})}{h^2} + \frac{h^8 D_i^{10}}{3150}, \tag{21}$$

where $D_i^9 = (\mathrm{d}^9 x/\mathrm{d}t^9)(\tau_i)$, $D_i^{10} = (\mathrm{d}^{10}x/\mathrm{d}t^{10})(\xi_i)$, $t_{i-4} < \tau_i, \xi_i < t_{i+4}$. Substitution of (20), (21) in (19), multiplication by h^2 and neglecting the discretization errors, yields the system

$$\begin{aligned} G_i(\boldsymbol{x}) = {} & \left(\frac{ch}{280} - \frac{1}{560}\right)x_{i-4} + \left(-\frac{4ch}{105} + \frac{8}{315}\right)x_{i-3} + \left(\frac{ch}{5} - \frac{1}{5}\right)x_{i-2} \\ & + \left(-\frac{4ch}{5} + \frac{8}{5}\right)x_{i-1} - \frac{205}{72}x_i + \left(\frac{ch}{280} - \frac{1}{560}\right)x_{i+4} + \left(\frac{4ch}{105} + \frac{8}{315}\right)x_{i+3} \\ & + \left(-\frac{ch}{5} - \frac{1}{5}\right)x_{i+2} + \left(\frac{4ch}{5} + \frac{8}{5}\right)x_{i+1} + h^2[F(x_i, t_i)] = 0 \quad \text{for } i \in \{1, \dots, N\}, \end{aligned} \tag{22}$$

where $\boldsymbol{x} = (x_1, \dots, x_N)$. We see that dropping the error terms in this case is equivalent to making an evaluation error on F of $\mathrm{O}(h^8)$, if the constants M_9 and M_{10} exist such that, $\forall \tau, \xi \in [0, T]$,

$$\left|\frac{\mathrm{d}^9 x}{\mathrm{d}t^9}(\tau)\right| \leqslant M_9 \quad \text{and} \quad \left|\frac{\mathrm{d}^{10} x}{\mathrm{d}t^{10}}(\xi)\right| \leqslant M_{10}.$$

Remarks. (1) Let us define, $\forall t \in [0, T]$,

$$\begin{aligned} \mathscr{L}_1(x(t)) = {} & \left(\frac{ch}{280} - \frac{1}{560}\right)x(t-4h) + \left(-\frac{4ch}{105} + \frac{8}{315}\right)x(t-3h) + \left(\frac{ch}{5} - \frac{1}{5}\right)x(t-2h) \\ & + \left(-\frac{4ch}{5} + \frac{8}{5}\right)x(t-h) - \frac{205}{72}x(t) + \left(\frac{4ch}{5} - \frac{8}{5}\right)x(t+h) \\ & + \left(-\frac{ch}{5} - \frac{1}{5}\right)x(t+2h) + \left(\frac{4ch}{105} + \frac{8}{315}\right)x(t+3h) + \left(-\frac{ch}{280} - \frac{1}{560}\right)x(t+4h), \end{aligned}$$

then it is clear that on the grid $\bar{\omega}_h$: $G_i(\boldsymbol{x}) = \bar{\mathscr{L}}_1(x(t_i)) + h^2 F(x_i, t_i)$.

(2) Applying the Newton method directly to (11),

$$\ddot{x}^{(n+1)} + c\dot{x}^{(n+1)} + F(x^{(n)} + \delta^{(n)}, t) = 0,$$

where $\delta^{(n)} = x^{(n+1)} - x^{(n)}$, yields

$$\ddot{x}^{(n+1)} + c\dot{x}^{(n+1)} + F(x^{(n)}, t) + \frac{\partial F}{\partial x}(x^{(n)}, t)\cdot\delta^{(n)} = 0$$

or finally

$$\ddot{x}^{(n+1)} + c\dot{x}^{(n+1)} + \frac{\partial F}{\partial x}(x^{(n)}, t)\cdot x^{(n+1)} = \frac{\partial F}{\partial x}(x^{(n)}, t)\cdot x^{(n)} - F(x^{(n)}, t).$$

Discretization of this equation, using the differentiation formulae (20), (21), will also give us the system (22).

If we apply the Newton method to this nonlinear sytem of N equations in N unknowns, we will want to solve a sequence of linear systems of the form

$$J^{(n)}(\boldsymbol{x}^{(n+1)} - \boldsymbol{x}^{(n)}) = -G(\boldsymbol{x}^{(n)})$$

or

$$J^{(n)}\boldsymbol{x}^{(n+1)} = B^{(n)} \equiv J^{(n)}\boldsymbol{x}^{(n)} - G(\boldsymbol{x}^{(n)}), \tag{23}$$

where $J^{(n)} = G'(\boldsymbol{x}^{(n)})$. Equation i of this system,

$$\sum_{j=-4}^{4} J_{i,P(i+j)} x_i^{(n)} = B_i^{(n)}, \tag{24}$$

contains only 9 nonzero matrix elements of $J^{(n)}$:

$$J_{i,P(i-4)} = \frac{ch}{280} - \frac{1}{560},$$

$$J_{i,P(i-3)} = -\frac{4ch}{105} + \frac{8}{315},$$

$$J_{i,P(i-2)} = \frac{ch}{5} - \frac{1}{5},$$

$$J_{i,P(i-1)} = -\frac{4ch}{5} + \frac{8}{5},$$

$$J_{i,P(i)} = -\frac{205}{72} + h^2\frac{\partial F}{\partial x_i},$$

$$J_{i,P(i+1)} = \frac{4ch}{5} + \frac{8}{5},$$

$$J_{i,P(i+2)} = -\frac{ch}{5} - \frac{1}{5},$$

$$J_{i,P(i+3)} = \frac{4ch}{105} + \frac{8}{315},$$

$$J_{i,P(i+4)} = -\frac{ch}{280} - \frac{1}{560},$$

where $P(k) = (k-1) \bmod N + 1$, and (23) has a second member:

$$B_i^{(n)} = \sum_{j=i-4}^{i+4} J_{i,P(j)} x_j^{(n)} - \left[\left(\sum_{j=i-4}^{i+4} J_{i,P(j)} x_j^{(n)} - h^2 \frac{\partial F}{\partial x}(x_i, t_i) x_i^{(n)} \right) + h^2 F(x_i, t_i) \right]$$
$$= h^2 \left[\frac{\partial F}{\partial x}(x_i, t_i) x_i^{(n)} - F(x_i, t_i) \right].$$

Starting from an initial solution vector $\boldsymbol{x}^{(0)}$ (obtained for instance from the solution of the given differential equation for another parameter value), we see that iteration (23) corresponds to solving a linear system with a banded matrix with the following structure,

$$\left(\begin{array}{ccccccccccccc}
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\times & \times & \times & \times & & & & & \times & \times & \times & \times & \times
\end{array}\right)$$

of bandwidth 9, for which only the diagonal elements need to be recalculated at each step. Using the Woodbury formula [15, pp. 69–70], the solution of this system can be reduced to solving six linear systems with the same 9-diagonal matrix and one linear system of five equations in five unknowns. To minimize computations, the six large 9-diagonal systems were solved simultaneously and, in order to keep the condition number of their matrix a constant, they were solved using a variant of the QR-factorization especially adapted to banded matrices, based on Givens rotations [9, p.156].

In all cases, where the multipliers of the corresponding homogeneous linear differential equation were different from one, the approximate solution $\boldsymbol{x}$ was obtained in a few steps, as was to be expected from the quadratic convergence of the Newton method.

3.2. Determination of the multipliers corresponding to a periodic solution

We transformed the variational equation

$$\ddot{\delta} + c\dot{\delta} + \frac{\partial F}{\partial x}(\bar{x}, t) = 0$$

of (11) into the normal system

$$\begin{aligned} \dot{\delta} &= \eta, \\ \dot{\eta} &= -\frac{\partial F}{\partial x}(\bar{x}, t)\,\delta - c\eta, \end{aligned} \tag{25}$$

which was integrated over the period T, using the initial conditions (1,0) and (0,1), in order to obtain the fundamental matrix $\Phi(T)$. By the Floquet theory, the multipliers of (25) are given by the eigenvalues of this matrix.

Note that Eq. (25) contains the approximate solution $\bar{x}$ of (11). For the numerical integration of (25) we used the Runge–Kutta–Hůta method [14]. As in this method the second members of (25) (and so the values of $\bar{x}$) need to be calculated at each step for $t_0 = t_i$, $t_1 = t_i + h/9$, $t_2 = t_i + h/6$, $t_3 = t_i + h/3$, $t_4 = t_i + h/2$, $t_5 = t_i + 2h/3$, $t_6 = t_i + 5h/6$, $t_7 = t_i + h = t_{i+1}$, we see that for t_j, $j \in \{1, \ldots, 6\}$, $\bar{x}$ needs to be interpolated between the already known values $\bar{x}_i$.

One could use Lagrangian interpolation of sufficiently high order with interpolation points

$$t_{i-k}, t_{i-k+1}, \ldots, t_i, t_{i+1}, \ldots, t_{i+k+1},$$

e.g., $k = 3$ yielding an error of order $O(h^8)$ (cf. [1, p. 879]), but we obtained better results by trigonometric interpolation:

$$\bar{x}(t) = \frac{a_0}{2} + \sum_{i=1}^{K} (a_i \cos i\omega t + b_i \sin i\omega t),$$

where $\omega = 2\pi/T$, $N = 2K + 1$. If $\bar{x}(t_i) = \bar{x}_i$ ($i \in \{1, \ldots, N\}$), we can write, $\forall t \in [0, T]$,

$$\bar{x}(t) = \sum_{i=1}^{N} \bar{x}_i \prod_{\substack{j \neq i \\ j=1}}^{N} \frac{\sin(\frac{1}{2}\omega(t - t_j))}{\sin(\frac{1}{2}\omega(t_i - t_j))}$$

(cf. [1, p. 881]) or

$$\bar{x}(t) = \frac{1}{N} \sum_{i=1}^{N} \bar{x}_i \frac{\sin(\frac{1}{2}N\omega(t - t_i))}{\sin(\frac{1}{2}\omega(t - t_i))} \tag{26}$$

or, alternatively,

$$\bar{x}(t) = \frac{\sin(\frac{1}{2}N\omega t)}{N} \sum_{i=1}^{N} \bar{x}_i \frac{(-1)^{i-1}}{\sin(\frac{1}{2}\omega(t - t_i))}.$$

Direct derivation of formula (26) also allows the determination of $\dot{x}$ and $\ddot{x}$ $\forall t \in [0, T]$:

$$\dot{x}(t) = \frac{\omega}{2N} \sum_{i=1}^{N} \bar{x}_i \left[N \frac{\cos(\frac{1}{2}N\omega(t-t_i))}{\sin(\frac{1}{2}\omega(t-t_i))} - \frac{\cos(\frac{1}{2}\omega(t-t_i))\sin(\frac{1}{2}N\omega(t-t_i))}{\sin^2(\frac{1}{2}\omega(t-t_i))} \right], \tag{27}$$

$$\ddot{x}(t) = \frac{\omega^2}{4N} \sum_{i=1}^{N} \bar{x}_i \left[\frac{(1+\cos^2(\frac{1}{2}\omega(t-t_i)))\sin(\frac{1}{2}N\omega(t-t_i))}{\sin^3(\frac{1}{2}\omega(t-t_i))} \right.$$

$$\left. - 2N \frac{\cos(\frac{1}{2}\omega(t-t_i))\cos(\frac{1}{2}N\omega(t-t_i))}{\sin^2(\frac{1}{2}\omega(t-t_i))} - N^2 \frac{\sin(\frac{1}{2}N(\omega(t-t_i))}{\sin(\frac{1}{2}\omega(t-t_i))} \right]. \tag{28}$$

For phase plot calculations the values of the derivatives in t_p, $p \in \{1, \dots, N\}$, can then be calculated by letting $t \to t_p$ in (27):

$$\dot{x}_p \equiv \dot{x}(t_p) = \frac{\omega}{2} \sum_{\substack{i \neq p \\ i=1}}^{N} \bar{x}_i \frac{(-1)^{p-i}}{\sin((p-i)\pi/N)}.$$

3.3. Continuation of the solution — determination of the bifurcations

(1) In all the examples that were treated, the starting solution for a new parametric value was simply the solution obtained for the preceding value of the system parameter. We found that it was possible to go from one bifurcation (with multiplier $M \lessapprox 1$) to the next one (with $M = -1$) in only a few steps.

(2) For the accurate determination of the bifurcation value of the system parameter w:

(a) We first had to bracket the root of the equation

$$M(w) + 1 = 0, \tag{29}$$

which was done using a search in the direction of the decreasing values of M.

(b) We then essentially used the Brent method [15, p.251]), a combination of root bracketing, bisection and inverse quadratic interpolation, to determine the value of the root of (29) with an accuracy of 10 digits.

(3) To obtain a stable $2^{n+1}T$-periodic bifurcate solution, we:

(a) Constructed an unstable 2^nT-periodic solution with M slightly smaller than -1.

(b) Determined initial conditions from this solution which, slightly perturbed, were used to integrate the nonlinear differential equation (1) a number of times over the double period, using the Runge–Kutta–Hůta method. As soon as the convergence to the stable solution was initiated, the last approximation evaluated on the given mesh was used as a starting solution in the linear system (23).

(c) In a few iterations the period doubled stable solution could then be obtained.

4. Existence of the periodic solution and error estimation

4.1. *The solution of* (22)

If we iterate on the system of equations (24) until convergence to the fixed point $\bar{x} = x^{(n)} \approx x^{(n-1)}$ (almost to machine precision), the solution found solves the system of equations (22) approximately. The existence of the solution of this system and the error estimation can then be derived from Proposition 2 in Section 2.2.

To apply this theorem, we need to calculate:

(1) $r \geqslant |F(\bar{\alpha})| = |G(\bar{x})|.$

This can be done by substitution of $\bar{\alpha} = \bar{x}$ in (22) and evaluation of $r = \sqrt{N}|G(\bar{x})|_\infty \geqslant |G(\bar{x})$.

(2) $M' \geqslant \|J^{-1}(\bar{\alpha})\| = \|J^{(n)-1}(\bar{x})\|.$

The parameter M' can be determined if we evaluate the square root of the largest eigenvalue of the symmetric matrix $A = (J^{(n)-1})^{\mathrm{T}}\, J^{(n)-1} = (J^{(n)} J^{(n)\mathrm{T}})^{-1}$. We approximate this eigenvalue by the following iteration:

(a) Initialization: Choose a vector $v^{(0)} = (1, 0, \ldots, 0) \in \mathbb{R}^N$.
(b) For $k = 1, \ldots$:
  (i) First solve $J^{(n)} y^{(k)} = v^{(k-1)}$.
  (ii) Then solve $J^{(n)\mathrm{T}} x^{(k)} = y^{(k)}$, and normalize $v^{(k)} = x^{(k)}/|x^{(k)}|_\infty$.
  (iii) If $\| x^{(k)}|_\infty - |x^{(k-1)}|_\infty|$ is small enough then stop, else increment k and go to (i).
(c) Finally: Take $\|J^{(n)-1}(\bar{x})\| = \sqrt{|x^{(k)}|_\infty}$.

Note. The linear systems that must be solved in steps (i) and (ii) have a banded matrix equal to the last system matrix $J^{(n)}$ used for the determination of $\bar{x}$ and its transpose. We also remark that this time, the diagonal need *not* be refreshed in each step.

(3) $\|J(\alpha) - J(\bar{\alpha})\| = \|J^{(n)}(x) - J^{(n)}(\bar{x})\|.$

In this case the difference of the two matrices above yields a diagonal matrix D with an ith diagonal element:

$$d_i = h^2\left[\frac{\partial F}{\partial x}(x_i) - \frac{\partial F}{\partial x}(\bar{x}_i)\right],$$

where $|x - \bar{x}| \leqslant \delta$. Thus we obtain

$$|d_i| = h^2\left|\frac{\partial^2 F}{\partial x^2}(\bar{x}_i)(x_i - \bar{x}_i) + \tfrac{1}{2}\frac{\partial^3 F}{\partial x^3}(\xi_i)(x_i - \bar{x}_i)^2\right|, \quad \xi_i \in [x_i, \bar{x}_i]$$

Table 1
Some classical Duffing-like equations found in the cited references

	a	b	c	d	$p(t)$
Huberman	1	-4	0	0.115	$\cos wt$
Higuchi	$-\frac{1}{2}$	$\frac{1}{2}$	0.24	0.5	$\sin wt$
Fang/Dowell	1	1	0	w	$\cos t$
Van Dooren/Janssen	1	1	3	w	$\cos t$

The symbol w represents the variable system parameter.

$$\leqslant h^2\left(\left|\frac{\partial^2 F}{\partial x^2}(\bar{x}_i)\right|\delta + \tfrac{1}{2}\left|\frac{\partial^3 F}{\partial x^3}(\xi_i)\right|\delta^2\right).$$

In Duffing-like equations (see [10, 11, 7, 26]), F is of the form

$$ax + bx^3 - c - dp(t),$$

where a, b, c, d and $p(t)$ are defined in Table 1.

In all these cases we finally obtain

$$|d_i| \leqslant 3h^2|b|\delta(2|\bar{x}_i| + \delta) \leqslant 3h^2|b|\delta(2|\bar{\boldsymbol{x}}|_\infty + \delta).$$

So, for the diagonal matrix D, condition (9) of Theorem 2 could be written as

$$\|J(\boldsymbol{x}) - J(\bar{\boldsymbol{x}}\| \leqslant \Delta(\delta) = 3h^2|b|\delta(2|\boldsymbol{x}|_\infty + \delta) < \frac{\kappa}{M'}. \tag{30}$$

The graphical representation of conditions (9) and (10), given in Fig. 1, shows us that a nonempty intersection of the domains determined by these conditions is possible if the following equation has three real roots:

$$3h^2|b|M'\delta^3 + 6h^2|b||\bar{\boldsymbol{x}}|_\infty M'\delta^2 - \delta + M'r = 0. \tag{31}$$

A necessary and sufficient condition for this to happen is

$$q^3 + p^2 < 0, \tag{32}$$

with

$$\begin{aligned} q &= \tfrac{1}{3}a_1 - \tfrac{1}{9}a_2^2, \\ p &= \tfrac{1}{6}a_1 a_2 - \tfrac{1}{2}a_0 - \tfrac{1}{2}a_2^3 \end{aligned} \tag{33}$$

and

$$\begin{aligned} a_2 &= 2|\bar{\boldsymbol{x}}|_\infty, \\ a_1 &= -\frac{1}{3h^2|b|M'}, \\ a_0 &= \frac{r}{3h^2|b|} \end{aligned} \tag{34}$$

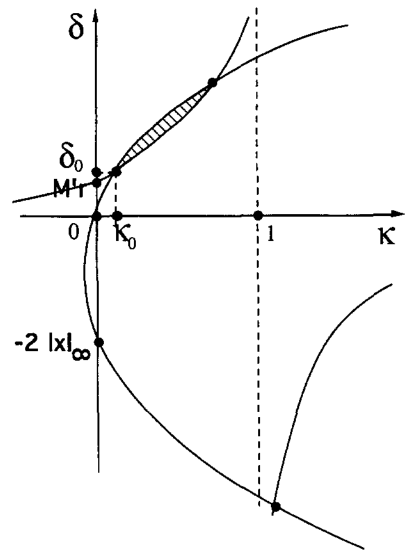

Fig. 1.

(cf. [1, p. 17]). As in this case, by (33), (34),

$$q = -\frac{1}{9h^2|b|M'} - \frac{4|\bar{x}|_\infty^2}{9} < 0$$

and

$$p = -\frac{|\bar{x}|_\infty}{9h^2|b|M'} - \frac{r}{6h^2|b|} - \frac{8|\bar{x}|_\infty^3}{27} < 0,$$

we can rewrite condition (32) in the form

$$-p = |p| < (-q)^{3/2},$$

which, by (33), (34) and solving for r, finally yields the condition

$$r < r_0 \equiv 6h^2|b|\left[\left(\frac{1}{9h^2|b|M'} + \frac{4|\bar{x}|_\infty^2}{9}\right)^{3/2} - \frac{|\bar{x}|_\infty}{9h^2|b|M'} - \frac{8|\bar{x}|_\infty^3}{27}\right]. \tag{35}$$

It can be shown that the factor between brackets in (35) is positive. If this condition is fulfilled, the smallest positive root of (31) will give us a value δ_0, which can be associated with $\kappa_0 = 1 - M'r/\delta_0$, indicating that an exact solution of (22) exists and that the error on the approximation is

$$E_1 = |\hat{x} - \bar{x}| \leqslant \delta_0 .$$

4.2. *The solution of* (1)

Here we wish to use Proposition 3 in the preceding section. We need to calculate:

(1) $r \geqslant \|\ddot{\bar{x}} + c\dot{\bar{x}} + F(\bar{x}, t)\|$.

We first could use (26) to determine a trignometric polynomial of degree K ($K = (N-1)/2$) that extends the discrete solution $\bar{x}_i \mid i \in \{1, \dots, N\}$ to the whole interval $[0, T]$. Then it is clear that the expression

$$\eta(t) \equiv \ddot{\bar{x}} + c\dot{\bar{x}} + F(\bar{x}, t) \tag{36}$$

is a trigonometric polynomial of degree $K' = 3K$ for the Duffing-like equations considered in Table 1. Putting $N' = 2K' + 1 = 3N - 3 + 1 = 3N - 2$, we can evaluate $\eta_i = \eta(t_i)$ with $t_i = (i-1)T/N'$, $i \in \{1, \dots, N'\}$, and obtain an explicit formula of (36) by

$$\eta(t) = \frac{1}{N'} \sum_{i=1}^{N'} \eta_i \frac{\sin(\frac{1}{2}N'\omega(t - t_i))}{\sin(\frac{1}{2}\omega(t - t_i))},$$

as well as of $\eta'(t)$, and $\eta''(t)$ using (27), (28). We are now able to calculate

$$r = \sup_{t \in [0,T]} \left| \frac{1}{N'} \sum_{i=1}^{N'} \eta_i \frac{\sin(\frac{1}{2}N'\omega(t - t_i))}{\sin(\frac{1}{2}\omega(t - t_i))} \right|.$$

(2) $$M_1 = \sqrt{2\pi \sup_{t \in [0,T]} \int_0^T H_{1,2}^2 \, \mathrm{d}s} \geqslant \|\mathscr{H}_1\|. \tag{37}$$

Bouc [2, 3] used an indirect method to evaluate the norm of the linear functional $\mathscr{H}_1$, associated with a given Galerkin approximation of order m, but he had to suppose that for some $m_0 \in \mathbb{N}$ all the Galerkin approximations of order $m \geqslant m_0$ exist. As in this application we had to solve the linear differential system (25) for the evaluation of the multipliers corresponding to the fundamental matrix $\Phi(T)$ associated with $\bar{x}$, we could as well save the intermediate values of the elements $\phi_1, \phi_1', \phi_2, \phi_2'$ of this matrix $\Phi(t_i)$ at $t_i \mid i \in (1, \dots, N+1)$ during the execution of the already mentioned Runge–Kutta–Hŭta method. Using (4) it is easy to show that, $\forall t_i$,

$$\begin{aligned} H_{1,2}^{\pm}(t_i, s) = e^{cs}[&- m_{1,1}^{\pm}\phi_1(t_i)\phi_2(s) + m_{1,2}^{\pm}\phi_1(t_i)\phi_1(s) \\ &- m_{2,1}^{\pm}\phi_2(t_i)\phi_2(s) + m_{2,2}^{\pm}\phi_2(t_i)\phi_2(s)], \end{aligned}$$

where

$$M^- = \begin{pmatrix} m_{1,1}^- & m_{1,2}^- \\ m_{2,1}^- & m_{2,2}^- \end{pmatrix} = (E - \Phi(T))^{-1} = \frac{\begin{pmatrix} 1 - \phi_2' & \phi_2 \\ \phi_1' & 1 - \phi_1 \end{pmatrix}}{1 - \operatorname{tr}\Phi}(T) \qquad \text{if } s < t_i,$$

$$M^+ = \begin{pmatrix} m_{1,1}^+ & m_{1,2}^+ \\ m_{2,1}^+ & m_{2,2}^+ \end{pmatrix} = (E - \Phi(T))^{-1}\Phi(T) = M^- - E \qquad \text{if } s \geqslant t_i.$$

Moreover, $H_{1,2}(t_i, s)$ being continuous and T-periodic in s, so is $H_{1,2}^2(t_i, s)$, and we could evaluate the integral in (37) with great precision using the trapezoidal sum in the Euler–Maclaurin summation formula (3.3.4) in [16]:

$$\int_0^T H_{1,2}^2(t_i,s)\,\mathrm{d}s = h\left[\sum_{j=1}^{i}{}'' H_{1,2}^{-2}(t_i,s_j) + \sum_{j=i}^{N+1}{}'' H_{1,2}^{+2}(t_i,s_j)\right] - h^{2m+2}\frac{B_{2m+2}T}{(2m+2)!}(H_{1,2}^{(2m+2)})^2(\xi),$$

$\xi \in [0, T]$.

assuming that the higher derivatives of $H_{1,2}^2(t_i, s)$ are well behaved and $h = T/N$ is small.[1]

(3) $$\left|\frac{\partial F}{\partial x}(x,t) - a(t)\right| = \left|\frac{\partial F}{\partial x}(x,t) - \frac{\partial F}{\partial x}(\bar{x})\right|.$$

Very much like in the discrete case, we could write

$$\begin{aligned}\left|\frac{\partial F}{\partial x}(x,t) - \frac{\partial F}{\partial x}(\bar{x},t)\right| &= \left|\frac{\partial^2 F}{\partial x^2}(\bar{x},t)(x-\bar{x}) + \tfrac{1}{2}\frac{\partial^3 F}{\partial x^3}(\xi,t)(x-\bar{x})^2\right|, \quad \xi \in [x, \bar{x}]\\ &\leqslant \left|\frac{\partial^2 F}{\partial x^2}(\bar{x},t)\right|\delta + \tfrac{1}{2}\left|\frac{\partial^3 F}{\partial x^3}(\xi,t)\right|\delta^2.\end{aligned}$$

Again, for the problems considered in Table 1, we could write condition 2 in Proposition 3 in the form

$$\left|\frac{\partial F}{\partial x}(x,t) - a(t)\right| \leqslant \Delta(\delta) = 3|b|\delta(2\|\bar{x}\| + \delta) \leqslant \frac{\kappa}{M_1}.$$

This last condition is analogous to condition (30) as it suffices to replace $(|\bar{x}|_\infty, M')$ by $(\|\bar{x}\|, M_1)$ and to put $h = 1$.

For the same reasons as in the previous subsection we now obtain the following existence condition:

$$r < \hat{r}_0 \equiv 6\,|b|\left[\left(\frac{1}{9|b|M_1} + \frac{4\|\bar{x}\|^2}{9}\right)^{3/2} - \frac{\|\bar{x}\|}{9|b|M_1} - \frac{8\|\bar{x}\|^3}{27}\right] \tag{38}$$

while the error estimate is given by

$$E_2 = \|\hat{x} - \bar{x}\| \leqslant \hat{\delta}_0$$

with $\hat{\delta}_0$ the smallest root of the equation

$$3|b|M_1\delta^3 + 6|b|\,\|\bar{x}\|M_1\delta^2 - \delta + M_1 r = 0.$$

---

[1] Here we have used the notation $\sum_{j=1}^{N+1}{}'' a_j \equiv a_1/2 + a_2 + \cdots + a_N + a_{N+1}/2$.

5. Numerical results

In the framework of the study of period doubling bifurcations leading to chaos, and in order to show the universal scaling property of Feigenbaum [8] in forced nonlinear oscillators, it is necessary to prove the existence of the exact solution of the given problems at those parametric values where bifurcation takes place, in other words, where one of the multipliers is equal to -1.

We applied the numerical methods explained in the preceding paragraphs in the fourth case from Table 1:

$$\ddot{x} + 0.1\dot{x} + x + x^3 = F_0 + F_1 \cos t. \tag{39}$$

In [26] with $F_0 = 3$, six Feigenbaum sequences were found in different ranges for the values of the parameter F_1, starting from first bifurcations $1T \to 2T$ at 14.4656, 24.2026, 31.8154, 46.9509, 83.8825, 98.2822 towards increasing values of F_1. In this paper we start from the first of these cascades and derive a number of bifurcations for the values of the second parameter $F_0 = 3.0, 2.8, 2.6, \ldots, 1.6$.

In Table 2 we list a number of period doubling bifurcations with respect to the parameter F_1, for the given range of values of F_0, and calculate:

(1) r_{approx}, the error on the approximate system of equations (22),
(2) r_0, the maximum admissible value for this error for the application of Proposition 2,
(3) δ_0, a maximum bound for the error E_1 on the solution of (22),
(4) r_{exact}, the error on the exact differential equation (1),
(5) $\hat{r}_0$, the maximum admissible value for this error for the application of Proposition 3,
(6) $\hat{\delta}_0$, a maximum bound for the error E_2 on the solution of (1), if $r_{\text{exact}} \leqslant \hat{r}_0$.

In Table 3 we recapitulate the bifurcation values of the parameter F_1 drawn from Table 2 and we list the estimated value of this parameter at the transition to chaos, obtained by Aitkin's Δ^2-method.

For the parameter values $F_0 = 3.0, 2.8, 2.6, 2.4, 2.2, 2.0, 1.8, 1.6$ a number of approximations of the Feigenbaum number, using formula

$$\delta_i = \frac{(F_1)_i - (F_1)_{2i}}{(F_1)_{2i} - (F_1)_{4i}}$$

(where $(F_1)_i = F_1$ such that the bifurcation $2^{i-1}T \to 2^i T$ takes place), were evaluated and tabulated in Table 4.

All the calculations were performed on a PC equipped with an Intel 80486 processor, using the double precission IEEE floating point type. The large bandwidth of the solutions to (39) limited the number of bifurcations that could be obtained in this case.

Table 2
Existence and error analysis of the given equation with F_0 ranging from 3 to 1.6

F_0	F_1	N	r_{approx}	r_0	δ_0	r_{exact}	$\hat{r}_0$	$\hat{\delta}_0$
3.0	14.4656	129	$7.0\cdot10^{-13}$	$5.8\cdot10^{-5}$	$2.1\cdot10^{-10}$	$3.5\cdot10^{-5}$	$7.5\cdot10^{-3}$	$4.5\cdot10^{-5}$
	15.6461	257	$1.1\cdot10^{-12}$	$9.9\cdot10^{-6}$	$7.8\cdot10^{-10}$	$1.2\cdot10^{-4}$	$8.0\cdot10^{-4}$	$4.5\cdot10^{-4}$
	15.8577	513	$6.6\cdot10^{-13}$	$7.2\cdot10^{-7}$	$1.6\cdot10^{-9}$	$1.6\cdot10^{-4}$	$4.7\cdot10^{-5}$	?
	15.9052	1025	$2.1\cdot10^{-12}$	$7.8\cdot10^{-8}$	$1.5\cdot10^{-8}$	$1.6\cdot10^{-4}$	$3.0\cdot10^{-6}$	?
	15.9155	2049	$3.6\cdot10^{-12}$	$9.8\cdot10^{-9}$	$7.6\cdot10^{-8}$	$1.7\cdot10^{-4}$	$2.1\cdot10^{-7}$	?
2.8	14.4922	129	$6.7\cdot10^{-13}$	$6.0\cdot10^{-5}$	$2.0\cdot10^{-10}$	$3.3\cdot10^{-5}$	$7.8\cdot10^{-3}$	$4.1\cdot10^{-5}$
	15.7135	257	$5.8\cdot10^{-13}$	$1.0\cdot10^{-5}$	$3.4\cdot10^{-10}$	$1.2\cdot10^{-4}$	$8.2\cdot10^{-4}$	$4.5\cdot10^{-4}$
	15.9159	513	$1.8\cdot10^{-12}$	$7.0\cdot10^{-7}$	$4.5\cdot10^{-9}$	$1.6\cdot10^{-4}$	$4.9\cdot10^{-5}$	?
	15.9613	1025	$1.8\cdot10^{-12}$	$7.9\cdot10^{-8}$	$1.3\cdot10^{-8}$	$1.7\cdot10^{-4}$	$3.2\cdot10^{-6}$	?
	15.9712	2049	$3.5\cdot10^{-13}$	$9.7\cdot10^{-9}$	$7.4\cdot10^{-9}$	$1.7\cdot10^{-4}$	$2.2\cdot10^{-7}$	?
2.6	14.5312	129	$1.2\cdot10^{-12}$	$6.2\cdot10^{-5}$	$3.5\cdot10^{-10}$	$3.0\cdot10^{-5}$	$8.1\cdot10^{-3}$	$3.7\cdot10^{-5}$
	15.7906	257	$4.3\cdot10^{-13}$	$1.0\cdot10^{-5}$	$2.9\cdot10^{-10}$	$1.2\cdot10^{-4}$	$8.3\cdot10^{-4}$	$4.5\cdot10^{-4}$
	15.9846	513	$2.9\cdot10^{-12}$	$6.9\cdot10^{-7}$	$7.5\cdot10^{-9}$	$1.6\cdot10^{-4}$	$5.1\cdot10^{-5}$	?
	16.0288	1025	$6.6\cdot10^{-12}$	$8.0\cdot10^{-8}$	$4.9\cdot10^{-8}$	$1.7\cdot10^{-4}$	$3.3\cdot10^{-6}$	?
	16.0383	2049	$3.1\cdot10^{-13}$	$9.8\cdot10^{-9}$	$6.5\cdot10^{-9}$	$1.7\cdot10^{-4}$	$2.4\cdot10^{-7}$	?
2.4	14.5828	129	$1.0\cdot10^{-12}$	$6.4\cdot10^{-5}$	$2.9\cdot10^{-10}$	$2.8\cdot10^{-5}$	$8.4\cdot10^{-3}$	$3.4\cdot10^{-5}$
	15.8719	257	$1.1\cdot10^{-12}$	$1.0\cdot10^{-5}$	$7.1\cdot10^{-10}$	$1.2\cdot10^{-4}$	$8.4\cdot10^{-4}$	$4.5\cdot10^{-4}$
	16.0568	513	$3.2\cdot10^{-12}$	$6.7\cdot10^{-7}$	$8.3\cdot10^{-9}$	$1.6\cdot10^{-4}$	$5.2\cdot10^{-5}$	?
	16.0996	1025	$1.8\cdot10^{-12}$	$8.0\cdot10^{-8}$	$1.4\cdot10^{-8}$	$1.7\cdot10^{-4}$	$3.4\cdot10^{-6}$	?
	16.1089	2049	$5.5\cdot10^{-13}$	$9.7\cdot10^{-9}$	$1.2\cdot10^{-8}$	$1.8\cdot10^{-4}$	$2.4\cdot10^{-7}$	?
2.2	14.6475	129	$4.1\cdot10^{-13}$	$6.7\cdot10^{-5}$	$1.1\cdot10^{-10}$	$2.5\cdot10^{-5}$	$8.7\cdot10^{-3}$	$3.0\cdot10^{-5}$
	15.9555	257	$1.8\cdot10^{-12}$	$1.0\cdot10^{-5}$	$1.2\cdot10^{-9}$	$1.2\cdot10^{-4}$	$8.3\cdot10^{-4}$	$4.7\cdot10^{-4}$
	16.1312	513	$1.5\cdot10^{-12}$	$6.5\cdot10^{-7}$	$3.8\cdot10^{-9}$	$1.7\cdot10^{-4}$	$5.1\cdot10^{-5}$	?
	16.1721	1025	$1.9\cdot10^{-12}$	$7.8\cdot10^{-8}$	$1.4\cdot10^{-8}$	$1.7\cdot10^{-4}$	$3.4\cdot10^{-6}$	?
	16.1810	2049	$2.1\cdot10^{-12}$	$9.5\cdot10^{-9}$	$4.4\cdot10^{-8}$	$1.8\cdot10^{-4}$	$2.4\cdot10^{-7}$	?
2.0	14.7277	129	$7.9\cdot10^{-14}$	$7.0\cdot10^{-5}$	$2.2\cdot10^{-11}$	$2.3\cdot10^{-5}$	$9.1\cdot10^{-3}$	$2.7\cdot10^{-5}$
	16.0427	257	$6.1\cdot10^{-13}$	$9.6\cdot10^{-6}$	$4.2\cdot10^{-10}$	$1.3\cdot10^{-4}$	$8.1\cdot10^{-4}$	$4.8\cdot10^{-4}$
	16.2096	513	$9.8\cdot10^{-13}$	$6.2\cdot10^{-7}$	$2.6\cdot10^{-9}$	$1.7\cdot10^{-4}$	$4.8\cdot10^{-5}$	?
	16.2483	1025	$2.0\cdot10^{-12}$	$7.5\cdot10^{-8}$	$1.6\cdot10^{-8}$	$1.7\cdot10^{-4}$	$3.2\cdot10^{-6}$	?
	16.2567	2049	$1.6\cdot10^{-12}$	$9.1\cdot10^{-9}$	$3.5\cdot10^{-8}$	$1.8\cdot10^{-4}$	$2.2\cdot10^{-7}$	?
1.8	14.8278	129	$2.5\cdot10^{-13}$	$7.3\cdot10^{-5}$	$6.8\cdot10^{-11}$	$2.1\cdot10^{-5}$	$9.5\cdot10^{-3}$	$2.4\cdot10^{-5}$
	16.1368	257	$1.7\cdot10^{-12}$	$9.1\cdot10^{-6}$	$1.2\cdot10^{-9}$	$1.3\cdot10^{-4}$	$7.9\cdot10^{-4}$	$4.9\cdot10^{-4}$
	16.2959	513	$6.0\cdot10^{-13}$	$5.8\cdot10^{-7}$	$1.7\cdot10^{-9}$	$1.7\cdot10^{-4}$	$4.4\cdot10^{-5}$	?
	16.3322	1025	$4.5\cdot10^{-12}$	$6.9\cdot10^{-8}$	$3.6\cdot10^{-8}$	$1.8\cdot10^{-4}$	$2.9\cdot10^{-6}$	?
	16.2567	2049	$2.6\cdot10^{-12}$	$8.4\cdot10^{-9}$	$5.9\cdot10^{-8}$	$1.8\cdot10^{-4}$	$2.0\cdot10^{-7}$	?
1.6	14.9596	129	$2.4\cdot10^{-12}$	$7.7\cdot10^{-5}$	$6.2\cdot10^{-10}$	$1.9\cdot10^{-5}$	$9.9\cdot10^{-3}$	$2.2\cdot10^{-5}$
	16.2428	257	$4.1\cdot10^{-13}$	$8.4\cdot10^{-6}$	$3.0\cdot10^{-10}$	$1.3\cdot10^{-4}$	$7.5\cdot10^{-4}$	$5.2\cdot10^{-4}$
	16.3955	513	$2.0\cdot10^{-12}$	$5.3\cdot10^{-7}$	$5.7\cdot10^{-9}$	$1.7\cdot10^{-4}$	$3.9\cdot10^{-5}$	?
	16.4296	1025	$4.6\cdot10^{-13}$	$6.3\cdot10^{-8}$	$3.8\cdot10^{-9}$	$1.7\cdot10^{-4}$	$2.6\cdot10^{-6}$	?
	16.4370	2049	$1.2\cdot10^{-12}$	$7.7\cdot10^{-9}$	$2.8\cdot10^{-8}$	$1.8\cdot10^{-4}$	$1.8\cdot10^{-7}$	?

The symbol ? in the last column means that the existence of an exact solution could not be proven.

Table 3
Convergence of the $(F_1)_i$ to $(F_1)_{chaos}$

F_0	3.0	2.8	2.6	2.4	2.2	2.0	1.8	1.6
$(F_1)_1$	14.4656	14.4922	14.5312	14.5828	14.6475	14.7277	14.8278	14.9596
$(F_1)_2$	15.6461	15.7135	15.7906	15.8719	15.9555	16.0427	16.1368	16.2428
$(F_1)_3$	15.8577	15.9159	15.9846	16.0568	16.1312	16.2096	16.2959	16.3955
$(F_1)_4$	15.9052	15.9613	16.0288	16.0996	16.1721	16.2483	16.3322	16.4296
$(F_1)_5$	15.9155	15.9712	16.0383	16.1089	16.1810	16.2567	16.2567	16.4370
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(F_1)_{chaos}$	15.9184	15.9739	16.0409	16.1115	16.1835	16.2590	16.3423	16.4390

Table 4
Convergence of the δ_i to the Feigenbaum number

F_0	3.0	2.8	2.6	2.4	2.2	2.0	1.8	1.6
δ_1	5.57765	6.03462	6.49367	6.97080	7.44493	7.87873	8.22609	8.40007
δ_2	4.45807	4.45143	4.37812	4.32463	4.29701	4.31659	4.38116	4.47644
δ_3	4.59544	4.61880	4.61607	4.60346	4.59458	4.59736	4.61158	4.63048
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\lim_{i\to\infty}\delta_i$	4.66920	4.66920	4.66920	4.66920	4.66920	4.66920	4.66920	4.66920

6. Concluding remarks

(1) Although software packages as described in [13, 5, 6] can be used to study the bifurcations of any one- or two-parameter nonlinear oscillator, the authors used a combination of a standard difference method, the well-known Newton method and a trigonometric interpolation method, in order to be able to apply the existence theorems from [18] and validate the solutions of nonlinear oscillators for the parametric values where bifurcation takes place.

(2) The application of a continuation method, based upon the ideas explained in Section 3.3, has shown that because of the large domain of attraction of the periodic solution vector $\bar{x}$, rather large increments can be given to the system parameter, to study its influence upon the multipliers of the variational system (25) without the risk of divergence as was the case in the shooting method [24, 12, 25, 13]. The method also yields results with comparable accuracy (if all the multipliers were different from 1), but they were obtained in a faster way.

(3) The study of the results of Table 2 reveals the existence of the solution to the approximate system of equations in all cases, while the existence of solution to the exact equation could only be proved for the first two bifurcations.

Although the calculations to prove the existence of a periodic solution in a neighbourhood of an approximate solution described in Section 4.2 are straightforward, it should be remembered that the conditions we apply are only sufficient conditions. This means that if an approximation does not satisfy Proposition 3 of Urabe [18] it is still possible that an exact solution exists in its neighbourhood.

The evidence of Table 2 shows that although the error r_{approx} on the system of equations (22) is small, the error r_{exact} on the exact equation (1) remains comparatively high. It seems quite possible that if we can improve on the trigonometric interpolation routine by another method that respects the periodic nature of the solution of these nonlinear equations, r_{exact} could be reduced and consequently condition (38) would be satisfied in more cases. In this respect it seems worthwhile to study the use of periodic splines. When this technique is used, it would be possible to adapt the mesh size in order to obtain a better global precision as in [5, 6].

Moreover, an analysis of the second member of inequality (38) also shows that a sharper evaluation of M_1 could have the same effect.

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